"Stochastic modelling and derivatives in traditional markets and in crypto-markets"

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Exercice 1 (Affine model - **Duffie, Kan (1996)**). Given a yield curve $\theta \mapsto R(t,\theta)$ decomposed by the affine function of the short term rate r_t as following:

$$R(t,\theta) = \frac{1}{\theta}(F(\theta) + G(\theta)r_t), \tag{1}$$

we want to prove that the equation satisfied by F, G are:

$$\begin{cases} F'(\theta) = \alpha_1 G(\theta) + \frac{1}{2}\beta_1 G(\theta)^2, \\ F(0) = 0, \end{cases} \qquad \begin{cases} G'(\theta) = \alpha_2 G(\theta) + \frac{1}{2}\beta_2 G(\theta)^2 - 1, \\ G(0) = 0, \end{cases} \tag{2}$$

and the short rate satisfies:

$$\mathrm{d}r_t = (\alpha_1 + \alpha_2 r_t)\mathrm{d}t + \sqrt{\beta_1 + \beta_2 r_t}\,\mathrm{d}W_t. \tag{3}$$

Notice that the Vasicek model is an example of this linear model.

1. Suppose the short rate r is an Itô process:

$$\mathrm{d}r_t = \mu(r_t)\mathrm{d}t + \sigma(r_t)\mathrm{d}W_t$$

   where μ, σ are Lipschitz continuous. Derive the PDE satisfied by the zero-coupon price $B(t,T)$.
   Hint: denote by $\mathcal{B}(t,r,T)$ the zero-coupon price function, identify decompositions from Ito formula and from self-financing condition.

2. Under the assumption of (1), suppose in addition μ, σ^2 are also affine,

$$\mu(x) = \alpha_1 + \alpha_2 x, \qquad \sigma^2(x) = \beta_1 + \beta_2 x.$$

   Deduce the ODEs satisfied by F, G w.r.t. α, β.

3. Identify the corresponding coefficients α, β in Vasicek model.

Another example is the Cox, Ingersoll et Ross model (1985):

$$\mu(x) = a(b - x), \qquad \sigma(x) = -\sigma\sqrt{x}.$$

Duffie and Kan (1996) proved that the short-term interest models of the form (3) are the only ones leading to affine interest rate curve (of the form (1)).
Other examples (without affine interest rate curve):

- Dothan model: $\mu(x) = ax, \sigma(x) = bx$.
- Courtadon model: $\mu(x) = a_1 x + a_2, \sigma(x) = bx$.

- Generalized log-normal model: $\mu(x,t) = a_1(t)x + a_2(t)x\log x, \sigma(t,x) = b(t)x$.

Exercice 2 (Option sur obligations - **Jamshidian (1989)**). On considère une obligation qui paye des coupons C_i aux dates $T_i, i \geq 1$ et rembourse le nominal N au temps T_n:

$$0 < T_1 < \cdots < T_n.$$

On se place dans un modèle de Vasicek.

- Donner le prix O_t de l'obligation à un temps $t < T_1$.
- On considère maintenant un call sur l'obligation à maturité $T \in [0, T_1)$, avec un strike K.

  Montrer que ce call sur obligation est égal à une somme de call sur zéro-coupons avec certains strikes bien choisis.